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THE LONGITUDINAL FREE VIBRATIONS OF A FIXED-FREE TWO-PHASE ELASTIC BAR

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1. Introduction

In this paper, the longitudinal free vibrations of a fixed-free bar are studied. It is assumed that the bar initially consists of two phases, one of which was obtained from the other by a martensitic phase transformation.[1] It is also assumed that both phases of the bar have elastic constitutive behavior. For a bar that consists entirely of one phase that behaves elastically, it is well known that during the free vibrations of the bar the displacement and stress at each point of the bar oscillate as time progresses [4]. If there is damping present, these oscillations will decay and go to zero as time goes to infinity, otherwise their amplitudes will remain constant in time. Considering this, for a bar that initially consists of two different phases that both behave elastically, one might expect that the displacement and stress at each point of the bar will also have oscillatory-type behavior during the free vibrations of the bar. If this is the case, the driving traction at the interface separating the two phases will oscillate. As a result of this, if the nominal phase boundary velocity is related to the driving traction through a kinetic relation that does not have an interval of the driving traction corresponding to a zero nominal phase boundary velocity, the nominal phase boundary velocity will also oscillate. Since energy is dissipated when the phase boundary moves and passes over particles of material of one phase converting them into particles of material of the other phase, one might conclude that the oscillatory-type response of the two-phase bar during the free vibrations of the bar should decay as time increases. It is this damping behavior of the two-phase bar that will be the main subject of this chapter. The solutions of the boundary value problem will be determined by a numerical method, and the damping of the two-phase bar will be

[1] Martensitic phase transformations are diffusionless solid-solid phase transformations which, among other things, have continuous displacements, with possible discontinuous strains, at their phase boundary. These types of transformations are also characterized by the product phase having a shape deformation relative to the undeformed parent phase, which corresponds to an unstressed undeformed configuration of that product phase. See [5] and [8] for more detailed and comprehensive discussions about these types of transformations.

studied as the material coefficients are varied. The values of the material coefficients that result in the maximum damping will also be investigated.

2. The Kinematics

A one-dimensional finite bar that initially consists of two phases is considered. It is assumed that the process under consideration occurs in a time interval $\Gamma = [t_0, t_1]$. Additionally, for the problem that is considered, a continuum model that was developed in [6] (see also [7]) is used.

Consider a stationary reference configuration R for the bar. Let x denote a point in R and let L be the length of the bar with respect to this reference configuration. Considering this, R can be expressed as $\mathrm{R} = \{x/\, x \in [0, L]\}$. Let $\hat{y}(x,t)$ be the suitably smooth and invertible mapping which maps R into the deformed configuration of the bar at each $t \in \Gamma$, with $\hat{y}(x,t) = x + \hat{u}(x,t)\ \forall\ (x,t) \in \mathrm{R} \times \Gamma$. The quantity $\hat{u}(x,t)$ represents the displacement of a particle of material at $y = \hat{y}(x,t)$ from the point $x \in \mathrm{R}$ at time $t \in \Gamma$. In the following, the two phases of the bar will be referred to as phase 1 and phase 2. Let $x = s(t)$ be the reference position of the phase boundary separating phase 1 from phase 2 at time $t \in \Gamma$. It is assumed that particles of material with reference points in $\mathrm{R}^- = \{x/\, x \in [0, s(t)]\}$ at time $t \in \Gamma$ are in phase 1, and it is assumed that particles of material with reference points in $\mathrm{R}^+ = \{x/\, x \in [s(t), L]\}$ at time $t \in \Gamma$ are in phase 2. It is assumed that R^- coincides with an unstressed undeformed configuration of phase 1. We next assume that there exists a shape deformation of phase 2 with respect to R^+ that corresponds to an unstressed undeformed configuration of that phase. Let R_1^+ be the reference configuration coinciding with this shape deformation for all $t \in \Gamma$. Let x_1 denote a point in R_1^+, and let $\tilde{x}_1(x,t)$ be the suitably smooth and invertible mapping which maps R^+ into R_1^+ at each $t \in \Gamma$, with $x_1 = \tilde{x}_1(x,t)\ \forall\ x \in \mathrm{R}^+$ at each $t \in \Gamma$. For the

following problem, it is assumed that $\tilde{x}_1$ is given by

$$\tilde{x}_1(x,t) = x + \gamma_0(x - s(t)). \tag{2.1}$$

The displacement gradient (transformation strain) corresponding to this choice of $\tilde{x}_1$ is γ_0, the Jacobian is $\tilde{J} = 1 + \gamma_0$, and $R_1^+ = \{x_1 / x_1 \in [\tilde{x}_1(s(t),t), \tilde{x}_1(L,t)]\} = \{x_1 / x_1 \in [s(t), L_1(t)]\}$, where $L_1(t) = \tilde{x}_1(L,t) = L + \gamma_0(L - s(t))$. It is also assumed that $\gamma_0 > -1$ so that reflections are excluded from (2.1). Additionally, the inverse $\tilde{x}$ of the $\tilde{x}_1$ given by (2.1) is

$$\tilde{x}(x_1,t) = \frac{1}{1+\gamma_0}(x_1 + \gamma_0 s(t)), \tag{2.2}$$

$\forall\, x_1 \in R_1^+$ at each $t \in \Gamma$. The mapping which maps R_1^+ into the deformed configuration of phase 2 at each $t \in \Gamma$ is represented by $\hat{y}_1(x_1,t)$, with $\hat{y}_1(x_1,t) = x_1 + \hat{u}_1(x_1,t)\ \forall\, x_1 \in R_1^+$ at each $t \in \Gamma$.

3. The Continuum Model

In the continuum model that is used here, the constitutive equations for each phase are defined with respect to different reference configurations. More specifically, the constitutive equations for phase 1 are defined with respect to R^-, and the constitutive equations for phase 2 are defined with respect to R_1^+. Additionally, the field equations for phase 1 are expressed with respect to R^-, and the field equations for phase 2 are expressed with respect to R_1^+ (see [6], [7]). The main advantage of using this continuum model for the problem under consideration is that the field equations are in forms that permit direct linearization. This is the case since the displacements for each phase are measured from a reference configuration coinciding with an unstressed undeformed configuration of that phase, and consequently, for

the appropriate boundary and initial conditions, the displacement gradients can be considered infinitesimal.[2]

4. The Field Equations and Jump Conditions

It is assumed that the process under consideration is a purely mechanical process with no body forces present. The general field equations and jump conditions using the type of continuum model described in the previous section and for a purely mechanical process were derived and discussed in [6], [7].

The field equations for the problem under consideration consist of the balance of linear momentum for phase 1 and the balance of linear momentum for phase 2. These equations are

$$\frac{\partial \sigma}{\partial x} = \rho a, \tag{4.1}$$

$\forall\, x \in \mathrm{R}^-$ at each $t \in \Gamma$, and

$$\frac{\partial \sigma_1}{\partial x_1} = \bar{\rho}_1 \bar{a}_1, \tag{4.2}$$

$\forall\, x_1 \in \mathrm{R}_1^+$ at each $t \in \Gamma$, respectively, where $\rho(x)$ is the density of the material per unit volume of R, $\bar{\rho}_1(x_1, t) = \rho(\tilde{x})/\tilde{J}$ is the density of phase 2 per unit volume of R_1^+, $\sigma(x,t)$ is the nominal stress with respect to R^- for phase 1, $\sigma_1(x_1,t)$ is the nominal stress with respect to R_1^+ for phase 2, $a(x,t) = \frac{\partial^2}{\partial t^2}\hat{y}(x,t)$, and $\bar{a}_1(x_1,t) = \left[\frac{\partial^2}{\partial t^2}\hat{y}_1(\tilde{x}_1(x,t),t)\right]_{\tilde{x}(x_1,t)}$.

As mentioned previously, the displacements at a phase boundary separating two phases involved in a martensitic phase transformation are continuous, while the strains may be discontinuous. Considering this, we require that $\hat{y}$ be continuous and the first

[2] It is also assumed here that the unstressed undeformed configuration of each phase corresponds to a relative minimum of the elastic potential for that phase.

and second derivatives of $\hat{y}$ be piecewise continuous on $R \times \Gamma$, with discontinuities occuring only at $x = s(t)$. The continuity of displacement condition in its most direct form is

$$y_1^+(s(t),t) = y^-(s(t),t)\,. \tag{4.3}$$

For the $\tilde{x}_1$ given by (2.1), (4.3) reduces to

$$\hat{u}_1^+(s(t),t) = \hat{u}^-(s(t),t). \tag{4.4}$$

Considering the continuity requirements on $\hat{y}$ and anticipating the form of the constitutive equations for the type of material under consideration, we require that the stress be piecewise continuous, with discontinuities occuring only at $x = s(t)$. The jump condition at $x = s(t)$ representing the balance of linear momentum is

$$\sigma_1^+ - \sigma^- + \rho(\bar{v}_1^+ - v^-)\dot{s} = 0, \tag{4.5}$$

where $\dot{s}(t) = \frac{ds(t)}{dt}$. Differentiating (4.3) with respect to time yields

$$(\gamma_0 + \gamma_1^+\gamma_0 + \gamma_1^+ - \gamma^-)\dot{s} + \bar{v}_1^+ - v^- = 0, \tag{4.6}$$

at $x = s(t)$, where $\gamma = \frac{\partial \hat{u}}{\partial x}$ and $\gamma_1 = \frac{\partial \hat{u}_1}{\partial x_1}$. Using (4.6) in (4.5), we can obtain an alternate form for the *linear momentum jump condition:*

$$\sigma_1^+ - \sigma^- = \rho(\gamma_0 + \gamma_1^+\gamma_0 + \gamma_1^+ - \gamma^-)(\dot{s})^2. \tag{4.7}$$

The remaining equation at the phase boundary is a kinetic relation relating $\dot{s}$ and the driving traction f (see [1]). This kinetic relation is a constitutive equation and will discussed in the next section. For the problem under consideration the driving traction is given by

$$f = \tilde{J}W_1^+ - W^- - \tfrac{1}{2}(\sigma_1^+ + \sigma^-)(\gamma_0 + \gamma_1^+\gamma_0 + \gamma_1^+ - \gamma^-). \tag{4.8}$$

We also have the boundary conditions at $x = 0$ and $x_1 = L_1(t)$ (or $x = L$). For the fixed boundary condition at $x = 0$, we have

$$u(0,t) = 0, \tag{4.9}$$

$\forall t \in \Gamma$. At the $x_1 = L_1(t)$ boundary, there is nothing applied to the boundary; i.e. it is a free boundary. Therefore, the traction at this boundary is necessarily zero. Thus, the boundary condition at $x_1 = L_1(t)$ is

$$\sigma_1(L_1(t),t) = 0, \tag{4.10}$$

$\forall\, t \in \Gamma$.

5. The Constitutive Equations

It is assumed that both phase 1 and phase 2 are homogeneous elastic (or hyperelastic) phases. In particular, for phase 1 we assume that there exists an elastic potential

$$W = W(\gamma), \tag{5.1}$$

such that

$$\sigma = \frac{\partial W}{\partial \gamma}, \tag{5.2}$$

and for phase 2 we assume that there exists an elastic potential

$$W_1 = W_1(\gamma_1), \tag{5.3}$$

such that

$$\sigma_1 = \frac{\partial W_1}{\partial \gamma_1}. \tag{5.4}$$

If we solve for $u_1 = \hat{u}_1(x_1, t)$ in the dynamic boundary value problem, the acceleration term $\bar{a}_1$ in Equation (4.2) will contain several inertial-type terms that are solely a result of $\tilde{x}_1$ being a function of time. This was discussed for the general three-dimensional problem in [6], [7]. As was further discussed there, but again for the general three-dimensional problem, these inertial-type terms can be avoided by instead solving for $u_1 = \bar{u}_1(x, t)$ in the boundary value problem, where $\bar{u}_1(x, t) = \hat{u}_1(\tilde{x}_1(x, t), t)$ and $\hat{u}_1(x_1, t) = \bar{u}_1(\tilde{x}(x_1, t), t)$. This will be done in the boundary value problem that is considered here.

It is assumed that phase 1 is unstressed at $\gamma = 0$ and phase 2 is unstressed at $\gamma_1 = 0$. We next assume that the initial conditions are such that $|\gamma| << 1 \; \forall \, x \in [0, s(t))$ and $|\gamma_1| << 1 \; \forall \, x \in (s(t), L]$, at each $t \in \Gamma$, where here and in the following, $\gamma_1 = \frac{\partial \bar{u}_1}{\partial x_1} = \frac{\partial \bar{u}_1}{\partial x} \frac{\partial \tilde{x}}{\partial x_1}$. For the $\tilde{x}$ given by (2.2), $\gamma_1 = \frac{1}{1+\gamma_0} \frac{\partial \bar{u}_1}{\partial x}$. For these assumptions, W for phase 1 can be written as

$$W = W^* + \tfrac{1}{2} E \gamma^2 + \mathrm{O}(\gamma^3), \tag{5.5}$$

where $W^* = W(0)$ and $E = \frac{\partial^2 W}{\partial \gamma^2}|_{\gamma=0}$, and W_1 for phase 2 can be written as

$$W_1 = W_1^* + \tfrac{1}{2} E_1 \gamma_1^2 + \mathrm{O}(\gamma_1^3), \tag{5.6}$$

where $W_1^* = W_1(0)$ and $E_1 = \frac{\partial^2 W_1}{\partial \gamma_1^2}|_{\gamma_1=0}$. From (5.2) and (5.5), the first-order approximation of σ for phase 1 is

$$\sigma = E\gamma = E \frac{\partial \hat{u}}{\partial x}, \tag{5.7}$$

and from (5.4) and (5.6) the first-order approximation of σ_1 for phase 2 is

$$\sigma_1 = E_1 \gamma_1 = \frac{E_1}{1 + \gamma_0} \frac{\partial \bar{u}_1}{\partial x}. \tag{5.8}$$

We note that the constitutive quantities for phase 1 given by (5.1), (5.2), (5.5), and (5.7) are defined with respect to R^-, and the constitutive quantities for phase 2 given by (5.3), (5.4), (5.6), and (5.8) are defined with respect to R_1^+, even though (5.6) and (5.8) are in terms of the independent variable $x \in \mathrm{R}^+$.

We next assume that $W^* = \tilde{J} W_1^*$. This assumption might be most appropriate if phase 1 and phase 2 represent two different variants of the same martensite. We note, however, that if this were the case we need not assume that $\tilde{E}_1 = \tilde{E}$, since in a real three-dimensional material the moduli in a given direction of two variants of martensite separated by a phase boundary are not, in general, the same, and this can be incorporated into a one-dimensional model by assuming that $\tilde{E}_1 \neq \tilde{E}$. For these assumptions and using (5.5)-(5.8), the first-order approximation of the driving traction given by (4.8) is

$$f = -\frac{1}{2}\left\{\frac{E_1}{1+\gamma_0}\left(\frac{\partial \bar{u}_1}{\partial x}\right)^+ + E\left(\frac{\partial \hat{u}}{\partial x}\right)^-\right\}\gamma_0, \tag{5.9}$$

We next postulate a kinetic (constitutive) relation

$$\dot{s} = \Phi(f), \tag{5.10}$$

such that

$$\Phi(f) f \geq 0, \tag{5.11}$$

for all f [1]. The requirement given by (5.11) is imposed so that energy is dissipated (or preserved if the equality sign holds) during a phase transformation, instead of being created. We further assume that for the problem under consideration, the first-order form of $\Phi(f)$ is

$$\Phi(f) = \frac{1}{\nu} f = -\frac{\gamma_0}{2\nu}\left\{ \frac{E_1}{1+\gamma_0}\left(\frac{\partial \bar{u}_1}{\partial x}\right)^{+} + E\left(\frac{\partial \hat{u}}{\partial x}\right)^{-} \right\}, \tag{5.12}$$

where the constant ν depends on the material and is such that $\nu \geq 0$ so that (5.11) is satisfied.

6. The Boundary Value Problem

Substituting (5.7) into (4.1) and expressing the acceleration in that equation in terms of $\hat{u}$, we obtain the following for the balance of linear momentum for phase 1:

$$E\frac{\partial^2 \hat{u}}{\partial x^2} = \rho \frac{\partial^2 \hat{u}}{\partial t^2}, \tag{6.1}$$

$\forall\, x \in (0, s(t))$ at each $t \in \Gamma$. Substituting (5.8) and $\bar{\rho}_1 = \rho/\tilde{J} = \rho/(1+\gamma_0)$ into (4.2), and calculating $\bar{a}_1$ in that equation using $\hat{y}_1(\tilde{x}_1(x,t),t) = \tilde{x}_1(x,t) + \bar{u}_1(x,t)$, we obtain the following for the balance of linear momentum for phase 2:

$$\frac{E_1}{(1+\gamma_0)}\frac{\partial^2 \bar{u}_1}{\partial x^2} = \rho\left(-\gamma_0 \frac{d^2 s}{dt^2} + \frac{\partial^2 \bar{u}_1}{\partial t^2}\right), \tag{6.2}$$

$\forall\, x \in (s(t), L)$ at each $t \in \Gamma$.

For the linearized problem, the continuity of displacement condition is still given by (4.4), since there is nothing to linearize in that equation. We next consider the linearized form of the linear momentum jump condition given by (4.7). Because $\dot{s}$ is related to γ and γ_1 through the kinetic relation given by (5.12), $\dot{s}^2$ is second-order in γ and γ_1; i.e $\dot{s}^2$ goes to zero at the same rate as γ^2, γ_1^2, and $\gamma\gamma_1$ go to zero. However, the second-order terms in $\dot{s}^2$ also contain the constants E^2/ν, E_1^2/ν, and EE_1/ν.

Considering the fact that the magnitudes of E and E_1 are very large, if the magnitude of ν is not also large, γ and γ_1 might have to be unrealistically small in order for the terms in $\dot{s}^2$ to have magnitudes that are negligible in comparison to the magnitudes of the first-order terms. In the following, it will be assumed that the values of the material coefficients are such that the magnitudes of the second-order terms in Equation (4.7) are negligible in comparison to the magnitudes of the first-order terms in that equation for realistic values of the infinitesimal strains.[3] The restrictions that this assumption puts on the relative values of ν and the other material coefficients can best be observed when the linear momentum jump condition and the kinetic relation are in nondimensional form, which will be done in the next section. The true first-order approximation of the linear momentum jump condition given by (4.7) is

$$\frac{E_1}{1+\gamma_0}\left[\frac{\partial \bar{u}_1}{\partial x}\right]_{(s(t),t)} - E\left[\frac{\partial \hat{u}}{\partial x}\right]_{(s(t),t)} = 0, \tag{6.3}$$

$\forall\, t \in \Gamma$, which is equivalent to the continuity of tractions across the interface. Using (6.3) in (5.12), the first-order approximation of the kinetic relation for the problem under consideration can be written as

$$\dot{s} = -\frac{\gamma_0 E}{\nu}\left[\frac{\partial \hat{u}}{\partial x}\right]_{(s(t),t)}, \tag{6.4}$$

$\forall\, t \in \Gamma$.

For the linearized problem, the fixed boundary condition is still given by (4.9), and using (5.8) in (4.10), the first-order approximation of the free boundary condition is

$$\left.\frac{\partial \bar{u}_1}{\partial x}\right|_{(L,t)} = 0, \tag{6.5}$$

$\forall\, t \in \Gamma$.

[3] We note that the values of the strains at the interface as time progresses are proportional to the initial conditions that are given.

7. The Nondimensional Form of the Boundary Value Problem

For the problem under consideration, we define the following nondimensional variables:

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \omega t, \quad \tilde{s}(\tilde{t}) = \frac{s(\tilde{t}/\omega)}{L},$$

$$\tilde{u}(\tilde{x},\tilde{t}) = \frac{\hat{u}(\tilde{x}L,\tilde{t}/\omega)}{L}, \quad \tilde{u}_1(\tilde{x},\tilde{t}) = \frac{\bar{u}_1(\tilde{x}L,\tilde{t}/\omega)}{L}, \tag{7.1}$$

$$\tilde{E} = \frac{E_1}{(1+\gamma_0)E}, \qquad \tilde{\nu} = \frac{\nu}{\gamma_0\sqrt{E\rho}},$$

where $\omega = \sqrt{E/(\rho L^2)}$.[4] Using these nondimensional variables, the nondimensional form of the balance of linear momentum for phase 1 is

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2}, \tag{7.2}$$

$\forall \tilde{x} \in (0, \tilde{s}(\tilde{t}))$ at each $\tilde{t} \in \tilde{\Gamma}$, where $\tilde{\Gamma} = [\tilde{t}_0, \tilde{t}_1] = [\omega t_0, \omega t_1]$, and the nondimensional form of the balance of linear momentum for phase 2 is

$$\tilde{E}\frac{\partial^2 \tilde{u}_1}{\partial \tilde{x}^2} + \gamma_0 \frac{d^2 \tilde{s}}{d\tilde{t}^2} = \frac{\partial^2 \tilde{u}_1}{\partial \tilde{t}^2}, \tag{7.3}$$

$\forall\, \tilde{x} \in (\tilde{s}(\tilde{t}), 1)$ at each $\tilde{t} \in \tilde{\Gamma}$.

The nondimensional form of the continuity of displacement condition is

$$\tilde{u}(\tilde{s}(\tilde{t}),\tilde{t}) = \tilde{u}_1(\tilde{s}(\tilde{t}),\tilde{t}), \tag{7.4}$$

$\forall\, \tilde{t} \in \tilde{\Gamma}$, and the nondimensional form of the first-order form of the balance of linear momentum jump condition given by (6.3) is

[4] In the following, $\tilde{x}$ will denote the nondimensional independent variable defined by $(7.1)_1$, and not the inverse function of the function $\tilde{x}_1$ which maps R^+ into R_1^+ at each $t \in \Gamma$.

$$\tilde{E}\left[\frac{\partial \tilde{u}_1}{\partial \tilde{x}}\right]_{(\tilde{s}(\tilde{t}),\tilde{t})} - \left[\frac{\partial \tilde{u}}{\partial \tilde{x}}\right]_{(\tilde{s}(\tilde{t}),\tilde{t})} = 0\,, \tag{7.5}$$

$\forall\, \tilde{t} \in \tilde{\Gamma}$. Additionally, the nondimensional form of the kinetic relation given by (6.4) is

$$\frac{d\tilde{s}}{d\tilde{t}} = -\frac{1}{\tilde{\nu}}\left[\frac{\partial \tilde{u}}{\partial \tilde{x}}\right]_{(\tilde{s}(\tilde{t}),\tilde{t})}, \tag{7.6}$$

$\forall \tilde{t} \in \tilde{\Gamma}$. In the previous section, the issue concerning when it is appropriate to neglect the second-order terms in the linear momentum jump condition was discussed. The nondimensional form of the lowest-order term that was neglected in that equation is $\gamma_0\left(\frac{d\tilde{s}}{d\tilde{t}}\right)^2$. From this, (7.6), and the definitions of the nondimensional parameters, we can observe what relative values of the material coefficients are appropriate for the assumption that the magnitude of $\gamma_0\left(\frac{d\tilde{s}}{d\tilde{t}}\right)^2$ is negligible in comparison to the magnitudes of the terms in (7.5), for realistic values of the initial conditions.

The nondimensional form of the fixed boundary condition is

$$\tilde{u}(0,\tilde{t}) = 0\,, \tag{7.7}$$

$\forall\, \tilde{t} \in \tilde{\Gamma}$, and the nondimensional form of the free boundary condition is

$$\left.\frac{\partial \tilde{u}_1}{\partial \tilde{x}}\right|_{(1,\tilde{t})} = 0\,, \tag{7.8}$$

$\forall\, \tilde{t} \in \tilde{\Gamma}$.

We must also specify initial conditions for the two-phase bar. In particular, for phase 1 we specify

$$\tilde{u}(\tilde{x},\tilde{t}_0) = \tilde{h}(\tilde{x}),$$

$$\left.\frac{\partial \tilde{u}}{\partial \tilde{t}}\right|_{(\tilde{x},\tilde{t}_0)} = \tilde{g}(\tilde{x}), \tag{7.9}$$

for $0 \leq \tilde{x} < \tilde{s}(\tilde{t})$, and for phase 2 we specify

$$\tilde{u}_1(\tilde{x},\tilde{t}_0) = \tilde{h}_1(\tilde{x}),$$

$$\left.\frac{\partial \tilde{u}_1}{\partial \tilde{t}}\right|_{(\tilde{x},\tilde{t}_0)} = \tilde{g}_1(\tilde{x}), \tag{7.10}$$

for $\tilde{s}(\tilde{t}) < \tilde{x} \leq 1$. We also must specify an initial position for the phase boundary. In particular, we specify

$$\tilde{s}(\tilde{t}_0) = \tilde{s}_0 . \tag{7.11}$$

The initial boundary value problem that will be considered consists of the field equations (7.2) and (7.3), the continuity of displacement condition (7.4), the linear momentum jump condition given by (7.5), the kinetic relation given by (7.6), the fixed boundary condition (7.7), the free boundary condition (7.8), and the initial conditions given by (7.9)-(7.11).

8. The Numerical Method of Solution

We can observe that the differential equations involving time derivatives in the boundary value problem presented in Section 7 consist of a wave equation given by

(7.2), a forced wave equation given by (7.3), and an ordinary differential equation given by (7.6). The boundary conditions for Equations (7.2) and (7.3), given by (7.7) and (7.8), are both with respect to fixed boundaries.[5] However, the jump conditions given by (7.4) and (7.5), which are also types of boundary conditions, are with respect to a moving boundary (i.e. the interface). We also note that because $\tilde{s}(\tilde{t})$ is one of the unknown dependent variables in the problem and the jump conditions are evaluated at $\tilde{x} = \tilde{s}(\tilde{t})$, the boundary value problem is inherently nonlinear with respect to $\tilde{s}(\tilde{t})$.

Dynamic boundary value problems involving moving interfaces within finite bodies have been studied before (see [3]). Some of the more well known of these problems are the class of problems considered to be Stephan problems. These types of problems involve melting solids, with a moving interface (or boundary) separating the solid from the liquid. The unknowns in these types of problems are the temperature distributions of both the liquid and solid phases, and the position of the interface separating these two phases. The governing equations for these Stephan problems consist of heat equations for both phases and an equation governing the motion of the interface. There are a variety of numerical methods that have been used to study these Stephan problems (see [3] for an overview and discussion of these methods). Among these numerical methods are several types of finite difference methods.

For the problem that is considered here, the type of numerical method that will be used is a finite difference method. This type of numerical method has been chosen, as apposed to, e.g., a finite element method, because it is probably the most straight forward to apply to the type of boundary value problem that is being considered. The particular finite difference method used here, however, does differ somewhat from the finite difference methods that have been used for the Stephan problems that are

[5] Note that if we were solving for $\hat{u}_1$ instead of $\tilde{u}_1$ in the boundary value problem, the free boundary condition would be with respect to a moving boundary.

discussed in [3]. Most of these differences reflect the fact that the field equations in the Stephan problems are of parabolic-type with monotonically decaying solutions, and the field equations in the problem considered here are of hyperbolic-type with decaying oscillatory solutions.

In the following, let $\mathcal{I}[a,b] = \{m \in \mathcal{Z} / \ a \leq m \leq b, \ \ a \in \mathcal{Z}, \ b \in \mathcal{Z}\}$, where $\mathcal{Z}$ denotes the set of all integers. For the finite difference method that is used here, the bar will be divided into n intervals, each of length $h = 1/n$. The points $x = ih, \ \ i \in \mathcal{I}[0,n]$, will be referred to as the nodes of the bar. In the finite difference method, the displacements at these nodes will be determined (i.e approximated). The time increment is denoted by T, and for convenience we assume in the following that $\tilde{t}_0 = 0$. Considering this, $\tilde{t} = jT, \ \ j \in \mathcal{I}[0,l_1]$, where $\tilde{t}_1$ and l_1 are chosen such that $l_1 T = \tilde{t}_1$. We let $s(j)$ represent $\tilde{s}(jT)$, and $k(j)$ denote the node such that $|s(j) - k(j)h| \leq h/2$, at time $\tilde{t} = jT$. Additionally, we let $p(j)h$ represent the distance from the node $k(j) - 1$ to $s(j)$, at time $\tilde{t} = jT$. Considering this, we can write $s(j)$ as $s(j) = [k(j) - 1 + p(j)]h$, at each $j \in \mathcal{I}[0,l_1]$. Let $u(i,j)$ represent $\tilde{u}(ih,jT)$, $0 \leq ih \leq s(j)$ at each $j \in \mathcal{I}[0,l_1]$, and let $u_1(i,j)$ represent $\tilde{u}_1(ih,jT)$, $s(j) \leq ih \leq 1$ at each $j \in \mathcal{I}[0,l_1]$. Additionally, let $q(j)$ denote the displacement at the interface at time $\tilde{t} = jT$; i.e. let $q(j)$ represent $\tilde{u}(\tilde{s}(jT),jT) = \tilde{u}_1(\tilde{s}(jT),jT)$. We also let $\Phi(j)$ represent the nondimensional kinetic relation given by (7.6) at time $\tilde{t} = jT$.

At each time increment, the numerical routine begins by calculating $s(j+1)$. We are given $s(0)$ as an initial condition. At $j = 0$, we use an Euler's method to approximate Equation (7.6) and obtain $s(1)$. In particular, we use

$$p(1) = p(0) + \frac{T}{h}\Phi(0) \tag{8.1}$$

to obtain $p(1)$. Since $k(0)$ is given when $s(0)$ is given, we can then obtain $s(1)$. For this case where $j = 0$, we can use the initial conditions given by $(7.9)_1$ and $(7.10)_1$ to obtain the derivative term in $\Phi(0)$. The specific form of these initial conditions will be discussed at a later point in this section. Also, we note that Euler's method has an $O(T)$ numerical error. At $j = 1$, we use an Adams-Bashforth two-step method, which has an $O(T^2)$ error (see [2]), to approximate Equation (7.6) and obtain $s(2)$. The resulting equation for $p(2)$ is

$$p(2) = p(1) + \frac{T}{2h}\{3\Phi(1) - \Phi(0)\}, \tag{8.2}$$

which can then be used to obtain $s(2)$. Also, to obtain the derivative term in $\Phi(1)$ we can use the initial conditions given by (7.9) and (7.10). For $j \in \mathcal{I}[2, l_1 - 1]$, we use an Adams-Bashforth three step method, which has an $O(T^3)$ error, to approximate Equation (7.6) and obtain $s(j+1)$. In particular, we use

$$p(j+1) = p(j) + \frac{T}{12h}\{23\Phi(j) - 16\Phi(j-1) + 5\Phi(j-2)\} \tag{8.3}$$

to obtain $p(j+1)$ for $j \in \mathcal{I}[2, l_1 - 1]$. $p(j+1)$ is then used to obtain $s(j+1)$. The specific form of the finite difference approximation in $\Phi(j)$, $j \in \mathcal{I}[2, l_1]$, will be discussed at a later point in this section. Also, because of the definition of $k(j+1)$, after each $p(j+1)$ is calculated, it must be checked to determine whether it is such that $0.5 \leq p(j+1) \leq 1.5$. If $p(j+1)$ is calculated to be such that $p(j+1) < 0.5$, $k(j+1)$ must be set to $k(j+1) = k(j) - 1$ and $p(j+1)$ must be updated to $p(j+1) \rightarrow p(j+1) + 1$. If $p(j+1)$ is calculated to be such that $p(j+1) > 1.5$, $k(j+1)$ must be set to $k(j+1) = k(j) + 1$ and $p(j+1)$ must be updated to $p(j+1) \rightarrow p(j+1) - 1$.

At each time step, once $p(j+1)$, $k(j+1)$, and $s(j+1)$ are determined, and $p(j+1)$ is updated if necessary, the displacements at the nodes are determined.

For each $j \in \mathcal{I}[1, l_1 - 1]$, the displacements $u(i, j+1)$, $i \in \mathcal{I}[1, k(j+1) - 1]$, and $u_1(i, j+1)$, $i \in \mathcal{I}[k(j+1)+1, n-1]$, are determined from centered-type difference equations. It will also be assumed in the following that $|s(j+1) - s(j)| < h/4$, $\forall\, j \in \mathcal{I}[0, l_1]$. This is done so that $u(i, j-1)$, $u(i, j)$, and $u(i, j+1)$ in the difference equations representing Equation (7.2) all correspond to phase 1, and $u_1(i, j-1)$, $u_1(i, j)$, and $u_1(i, j+1)$ in the difference equations representing Equation (7.3) all correspond to phase 2.[6]

To obtain the initial displacements at the nodes, we use the initial conditions given by $(7.9)_1$ and $(7.10)_1$. We assume that these initial displacements are continuous $\forall\, \tilde{x} \in (0, 1)$, satisfy the boundary conditions given by (7.7) and (7.8), are such that the linear momentum jump condition given by (7.5) is satisfied, and have a first mode type mode shape. In particular, we assume that

$$\tilde{h}(\tilde{x}) = \varepsilon_0 \tilde{x}\,, \tag{8.4}$$

for $0 \leq \tilde{x} \leq \tilde{s}(0)$, and

$$\tilde{h}_1(\tilde{x}) = \frac{\varepsilon_0}{2\tilde{E}(\tilde{s}(0) - 1)} \left\{ (\tilde{x} - 1)^2 - (\tilde{s}(0) - 1)^2 \right\} + \varepsilon_0 \tilde{s}(0)\,, \tag{8.5}$$

for $\tilde{s}(0) \leq \tilde{x} \leq 1$. The values of the displacements at the next time increment can be approximated by the Taylor series expansion in time of the displacements. In particular, using the initial conditions given by (7.9) and (7.10), the first-order approximation of $\tilde{u}$ and $\tilde{u}_1$ at $\tilde{t} = T$ are

$$\tilde{u}(\tilde{x}, T) = \tilde{h}(\tilde{x}) + \tilde{g}(\tilde{x}) T\,, \tag{8.6}$$

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[6] Note that this is consistent with the assumption that the magnitudes of the terms in $\dot{s}^2$ in the linear momentum jump condition are negligible in comparison to the magnitudes of the first-order terms in that jump condition.

for $0 \leq \tilde{x} \leq \tilde{s}(0)$, and

$$\tilde{u}_1(\tilde{x}, T) = \tilde{h}_1(\tilde{x}) + \tilde{g}_1(\tilde{x})T\,, \tag{8.7}$$

for $\tilde{s}(0) \leq \tilde{x} \leq 1$, respectively. It is assumed that the initial velocity distribution results in a continuous displacement at the phase boundary at time $\tilde{t} = T$, results in the boundary conditions given by (7.7) and (7.8) being satisfied at time $\tilde{t} = T$, results in the linear momentum jump condition given by (7.5) being approximately satisfied at time $\tilde{t} = T$, and has a first-mode type velocity profile. Such an initial velocity distribution is given by

$$\tilde{g}(\tilde{x}) = v_0\tilde{x}\,, \tag{8.8}$$

for $0 \leq \tilde{x} \leq \tilde{s}(0)$, and

$$\tilde{g}_1(\tilde{x}) = \frac{v_0}{2\tilde{E}(\tilde{s}(T)-1)}\left\{(\tilde{x}-1)^2 - (\tilde{s}(T)-1)^2\right\} + v_0\tilde{s}(T)\,, \tag{8.9}$$

for $\tilde{s}(0) \leq \tilde{x} \leq 1$. This initial velocity distribution was used in (8.6) and (8.7), which were then used to obtain the displacements at the nodes at $\tilde{t} = T$.[7]

For $j \in \mathcal{I}[1, l_1 - 1]$, $u(0, j+1)$ is obtained from the following equation which represents the fixed boundary condition given by (7.7):

$$u(0, j+1) = 0\,. \tag{8.10}$$

$\forall\ j \in \mathcal{I}[1, l_1 - 1]$. We can obtain $u(i, j+1)$, for $i \in \mathcal{I}[1, k(j+1) - 2]$ at each $j \in \mathcal{I}[1, l_1 - 1]$, from a finite difference approximation of Equation (7.2) which uses centered difference equations for equally spaced nodes to approximate each term in

[7] We note that $\tilde{s}(T)$ is not a given constant in the problem and therefore cannot be used if an analytical solution was to be obtained. In this case, we could use $\tilde{s}(0)$ instead of $\tilde{s}(T)$ in (8.9) as an approximation. For the numerical method, however, we can use $\tilde{s}(T)$.

that equation. In the following, centered difference equations for equally spaced nodes will be referred to as *standard* centered difference equations. The resulting difference equation that is used to obtain $u(i, j+1)$ is

$$u(i,j+1) = 2u(i,j) - u(i,j-1) + \alpha\{u(i+1,j) - 2u(i,j) + u(i-1,j)\}, \quad (8.11)$$

for $i \in \mathcal{I}[1, k(j+1)-2]$ at each $j \in \mathcal{I}[1, l_1 - 1]$, where $\alpha = (T/h)^2$. The difference equation given by (8.11) is commonly used for the wave equation [2]. It is also well known that a numerical routine using this difference equation is numerically stable if and only if $\alpha \leq 1$, and that the numerical error increases as α decreases from 1 [2]. We can obtain $u_1(i, j+1)$, for $i \in \mathcal{I}[k(j+1)+2, n-1]$ at each $j \in \mathcal{I}[1, l_1 - 1]$, from a finite difference approximation of Equation (7.3) which uses standard centered difference equations to approximate each term in that equation. The resulting difference equation that is used to obtain $u_1(i, j+1)$ is

$$\begin{aligned} u_1(i,j+1) = 2u_1(i,j) - u_1(i,j-1) + \tilde{E}\alpha\{u_1(i+1,j) - 2u_1(i,j) + u_1(i-1,j)\} \\ + \gamma_0\{s(j+1) - 2s(j) + s(j-1)\}, \end{aligned} \quad (8.12)$$

for $i \in \mathcal{I}[k(j+1)+2, n-1]$ at each $j \in \mathcal{I}[1, l_1 - 1]$. For $j \in \mathcal{I}[1, l_1 - 1]$, we can obtain $u(n, j+1)$ from an $O(h^2)$ difference equation representing the free boundary condition given by Equation (7.8). This difference equation is

$$u_1(n,j+1) = \frac{1}{3}\{4u_1(n-1,j+1) - u_1(n-2,j+1)\}, \quad (8.13)$$

for each $j \in \mathcal{I}[1, l_1 - 1]$.

For the displacements $u(k(j+1)-1, j+1)$ and/or $u_1(k(j+1)+1, j+1)$, finite difference methods using $q(j)$ will be used. For the derivation of the difference equations for these displacements, we first note that the distance separating $s(j)$

from its nearest nodes is not equal to h. Because of this, difference equations for unequally spaced nodes must be used to represent $\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}$ and/or $\frac{\partial^2 \tilde{u}_1}{\partial \tilde{x}^2}$. These difference equations will be derived from the second-degree Lagrange interpolating polynomials for $\tilde{u}$ and/or $\tilde{u}_1$ near the interface. For a function $g(x)$, its second-degree Lagrange polynomial, denoted by $P(x)$, is given by

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}g(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}g(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}g(x_2), \tag{8.14}$$

where x_0, x_1, x_2, $g(x_0)$, $g(x_1)$, and $g(x_2)$ are given (see [2]). The first derivative of $P(x) \approx g(x)$ is

$$\frac{dP(x)}{dx} = \frac{x-x_1+x-x_2}{(x_0-x_1)(x_0-x_2)}g(x_0) + \frac{x-x_0+x-x_2}{(x_1-x_0)(x_1-x_2)}g(x_1) + \frac{x-x_0+x-x_1}{(x_2-x_0)(x_2-x_1)}g(x_2), \tag{8.15}$$

and the second derivative is

$$\frac{d^2P(x)}{dx^2} = \frac{2}{(x_0-x_1)(x_0-x_2)}g(x_0) + \frac{2}{(x_1-x_0)(x_1-x_2)}g(x_1) + \frac{2}{(x_2-x_0)(x_2-x_1)}g(x_2). \tag{8.16}$$

If we choose $x = x_0$, x_1, or x_2 in Equations (8.15) and (8.16), the errors in these equations are approximately $O(d^2)$, where d is the maximum distance between any of these three points.[8] Also, we note that when x_0, x_1, and x_2 are equally spaced, (8.16) reduces to a standard centered difference equation of the form that was used in (8.11) and (8.12).

[8] See [2] for a more detailed discussion of the error that is involved in the Lagrange interpolating polynomial and its derivatives. Also, difference equations of this type are used in some finite difference methods for Stephan problems (see [3]).

Case I: Consider the case where $0.5 \leq p(j+1) \leq 1.5$, before being updated (Figure 1a). For this case, $k(j+1) = k(j)$, and for the calculation of $u(k(j+1)-1, j+1)$, a difference equation of the form (8.16) will be used to approximate $\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}$ at $\tilde{t} = jT$ in Equation (7.2). Using this difference equation and a standard centered difference equation for the second time derivative term in Equation (7.2), we obtain

$$u(k(j+1)-1, j+1) = 2u(k(j+1)-1, j) - u(k(j+1)-1, j-1)$$

$$+ 2\alpha \left\{ \frac{u(k(j)-2, j)}{1+p(j)} - \frac{u(k(j)-1, j)}{p(j)} + \frac{q(j)}{[1+p(j)]p(j)} \right\}. \tag{8.17}$$

Similarly, for the calculation of $u_1(k(j+1)+1, j+1)$, a difference equation of the form (8.16) will be used to approximate $\frac{\partial^2 \tilde{u}_1}{\partial \tilde{x}^2}$ at $\tilde{t} = jT$ in Equation (7.3). Using this difference equation and standard centered difference equations for the second time derivative terms in Equation (7.3), we obtain

$$u_1(k(j+1)+1, j+1) = 2u_1(k(j+1)+1, j) - u_1(k(j+1)+1, j-1)$$

$$+ 2\tilde{E}\alpha \left\{ \frac{q(j)}{[2-p(j)][3-p(j)]} - \frac{u_1(k(j)+1, j)}{2-p(j)} + \frac{u_1(k(j)+2, j)}{3-p(j)} \right\} \tag{8.18}$$

$$+ \gamma_0 \{ s(j+1) - 2s(j) + s(j-1) \}.$$

We note that when $p(j) = 1$, Equations (8.17) and (8.18) reduce to Equations (8.11) and (8.12), respectively.

Case II: Consider the case where $p(j+1) < 0.5$, before being updated (Figure 1b). For this case, $k(j+1) = k(j) - 1$ and $p(j+1) \rightarrow p(j+1) + 1$. Additionally, in this case, we can use the difference equation given by (8.11) for the calculation of $u(k(j+1)-1, j+1)$. However, for the calculation of $u_1(k(j+1)+1, j+1)$, a

difference equation of the form (8.16) will be used to approximate $\frac{\partial^2 \tilde{u}_1}{\partial \tilde{x}^2}$ and standard centered difference equations will be used to approximate the two time derivative terms in Equation (7.3). The resulting difference equation is

$$u_1(k(j+1)+1, j+1) = 2u_1(k(j+1)+1, j) - u_1(k(j+1)+1, j-1)$$

$$+ 2\tilde{E}\alpha\left\{\frac{q(j)}{[1-p(j)][2-p(j)]} - \frac{u_1(k(j),j)}{1-p(j)} + \frac{u_1(k(j)+1,j)}{2-p(j)}\right\} \tag{8.19}$$

$$+ \gamma_0\{s(j+1) - 2s(j) + s(j-1)\}.$$

Case III: The last case that can occur is $p(j+1) > 1.5$, before being updated (Figure 1c). In this case, $k(j+1) = k(j)+1$ and $p(j+1) \to p(j+1)-1$. In this case, for the calculation of $u(k(j+1)-1, j+1)$, a difference equation of the form (8.16) will be used to approximate $\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}$ and a standard centered difference equation will be used to approximate the second time derivative term in Equation (7.2). The resulting difference equation is

$$u(k(j+1)-1, j+1) = 2u(k(j+1)-1, j) - u(k(j+1)-1, j-1)$$

$$+ 2\alpha\left\{\frac{u(k(j)-1,j)}{p(j)} - \frac{u(k(j),j)}{p(j)-1} + \frac{q(j)}{p(j)[p(j)-1]}\right\}. \tag{8.20}$$

For this case, we can use the difference equation given by (8.12) for the calculation of $u_1(k(j+1)+1, j+1)$.

Once $s(j+1)$ and the displacements at the nodes $i \in \mathcal{I}[0, k(j+1)-1]$ and $i \in \mathcal{I}[k(j+1)+1, n]$ have been determined, we determine the displacement $q(j+1)$ at the interface from the difference equation representing the linear momentum jump

condition. Using difference equations of the form (8.15) to approximate the spatial derivatives in Equation (7.5), we obtain

$$q(j+1)=\left\{\frac{\tilde{E}[2p-5]}{[2-p][3-p]}-\frac{2p+1}{[1+p]p}\right\}^{-1}\left\{\frac{p}{1+p}u(k-2,j+1)\right.$$

$$-\frac{1+p}{p}u(k-1,j+1)-\tilde{E}\frac{3-p}{2-p}u_1(k+1,j+1) \qquad (8.21)$$

$$\left.-\tilde{E}\frac{p-2}{3-p}u_1(k+2,j+1)\right\},$$

where $k = k(j+1)$ and $p = p(j+1)$.

The last displacement that must be calculated at each time increment is the displacement at the node $k(j+1)$. This displacement will be calculated using a second-degree Lagrange interpolating polynomial. In particular, if $k(j+1)h < s(j+1)$, we calculate $u(k(j+1),j+1)$ from

$$u(k,j+1)=\frac{1-p}{1+p}u(k-2,j+1)+\frac{2[p-1]}{p}u(k-1,j+1)+\frac{2q(j+1)}{[1+p]p}, \qquad (8.22)$$

where $k = k(j+1)$ and $p = p(j+1)$, and if $k(j+1)h > s(j+1)$, we calculate $u_1(k(j+1),j+1)$ from

$$u_1(k,j+1)=\frac{2q(j+1)}{[2-p][3-p]}+\frac{2[1-p]}{2-p}u_1(k+1,j+1)+\frac{p-1}{3-p}u_1(k+2,j+1), \qquad (8.23)$$

where $k = k(j+1)$ and $p = p(j+1)$.

The last quantity that is calculated at each time step is $\Phi(j+1)$. Recall that this quantity is used in the calculation of $p(j+1)$ at the next time step. For the

calculation of $\Phi(j+1)$, $j \in \mathcal{I}[1, l_1 - 1]$, a finite difference equation of the form (8.15) is used for the approximation of $\frac{\partial \tilde{u}}{\partial \tilde{x}}\big|_{(\tilde{s}(\tilde{t}),\tilde{t})}$. The resulting finite difference equation for $\Phi(j+1)$ is

$$\Phi(j+1) = -\frac{1}{\tilde{\nu}h}\left\{\frac{p}{1+p}u(k-2,j+1) - \frac{1+p}{p}u(k-1,j+1) + \frac{2p+1}{[1+p]p}q(j+1)\right\}, \tag{8.24}$$

for $j \in \mathcal{I}[1, l_1 - 1]$, where $k = k(j+1)$ and $p = p(j+1)$.

The numerical routine that was discussed above allows for the phase boundary to pass over nodes other than the node $k(j)$. However, for a problem that uses a kinetic relation of the form given by (7.6), most values of the material coefficients that are consistent with the assumption that the second-order terms in the linear momentum jump condition are negligible in comparison to the first-order terms in that equation and most initial conditions that produce infinitesimal initial stains will result in the phase boundary staying within the interval between the nodes $k(j) - 1$ and $k(j) + 1$ for all $\tilde{t} \in \tilde{\Gamma}$. In this case, a simplified numerical routine can be used where $k(j)$ has the same value at each time increment, $p(j+1)$ never needs to be updated (in the sense that it was updated in Cases II and III), and only Case I for the calculation of the displacements near the interface needs to be considered.

9. The Free Vibrations and Damping Properties

In this section, the free vibrations of the two-phase bar that were determined from the finite difference method that was discussed in the previous section are discussed. As expected, the response of the two-phase bar to the initial conditions given by (8.4), (8.5), (8.8), (8.9), and (7.11) has a decaying oscillatory form. In

particular, the position of the phase boundary oscillates as time progresses and decays to a new position that has a distance and direction from its initial position that is proportional to the magnitude and "direction" of its initial conditions (Figures 2–4). The displacements also have a decaying oscillatory form, and they go to zero as time goes to infinity (Figures 5–9). The mode shape of the bar during these free vibrations has a first mode type form (Figure 10)[9]. This is most likely a result of the fact that first mode type initial conditions were given.

9.1. The Damping Behavior

The damping of the bar was studied as $\tilde{\nu}$, $\tilde{E}$, and γ_0 were varied. In Figure 11, a plot of the settling time versus $\tilde{\nu}$ is presented.[10] Here, the settling time is defined as the nondimensional time necessary for the amplitude of $\tilde{u}_1(L,t)$ to become less than 10^{-4}. As the settling time decreases, it is said that the damping of the bar increases. As expected, as $\tilde{\nu}$ decreases, the damping of the bar increases. This is primarily a result of the fact that for a given amount of strain at the interface, the nominal phase boundary velocity increases as $\tilde{\nu}$ decreases. Consequently, as $\tilde{\nu}$ decreases, there is more motion of the phase boundary in a given interval of time, which results in more energy being dissipated in that interval of time. This increase in damping as $\tilde{\nu}$ decreases is also displayed in Figures 2–7. From Figure 11, it appears that the "frequency" of oscillation does not significantly depend on $\tilde{\nu}$. In particular, it can be observed from this figure that as $\tilde{\nu}$ is varied the settling time remains constant over an interval of $\tilde{\nu}$, and when the settling time changes, it does so discontinuously. This

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[9] In Figure 10, $\tilde{u}$ vs $\tilde{t}$ is plotted for $0 \le \tilde{x} \le \tilde{s}(\tilde{t})$ and $\tilde{u}_1$ vs $\tilde{t}$ is plotted for $\tilde{s}(\tilde{t}) \le \tilde{x} \le 1$. The shape deformation for phase 2 is not plotted.

[10] In this figure, the settling time versus $\tilde{\nu}$ is not presented for $0 < \tilde{\nu} < 0.09$ because the numerical routine is unstable for these values of $\tilde{\nu}$ when $\tilde{E} = 1.15$, $\gamma_0 = 0.1$, $T = 0.01$, and $h = 0.02$. This will be discussed further in the next section. Also, in this figure and in Figures 12 and 13, the initial conditions that were used are $e_0 = 0.001$, $v_0 = 0.001$, and $s_0 = 0.5$.

is a reflection of the fact that as $\tilde{\nu}$ decreases the amplitude of oscillation decreases, but the "frequency" of oscillation does not.

In Figure 12, the settling time versus $\tilde{E}$ is presented. From this figure it can be concluded that as $\tilde{E}$ goes to zero, the damping goes to zero (i.e. the settling time goes to *infinity*), and as $\tilde{E}$ increases, the damping increases. Additionally, this increase in damping levels off at a relatively high level of damping (i.e. at a relatively short settling time) after $\tilde{E} \approx 0.15$.

For the plots of the settling time versus the transformation strain γ_0, we do not vary γ_0 and keep $\tilde{E}$ and $\tilde{\nu}$ constant. This is because $\tilde{E}$ and $\tilde{\nu}$ can remain constant as γ_0 is varied only if E_1/E and $\nu/\sqrt{\rho E}$ also change values. We instead let $\tilde{E}' = E_1/E$ and $\tilde{\nu}' = \nu/\sqrt{\rho E}$ and substitute $\tilde{E} = \tilde{E}'/(1+\gamma_0)$ and $\tilde{\nu} = \tilde{\nu}'/\gamma_0$ into the difference equations of the finite difference program. We then plot the settling time versus γ_0 *while keeping* $\tilde{E}'$ *and* $\tilde{\nu}'$ *constant.* An example of such a plot is shown in Figure 13. From this figure we can conclude that as γ_0 goes to zero the damping goes to zero, and as the magnitude of γ_0 increases the damping increases. We can also observe from this figure that as the magnitude of γ_0 becomes greater than $|\gamma_0| \approx 0.06$ the damping becomes appreciable, and for $|\gamma_0|$ less than this value the damping is relatively small. It is interesting to note that if one were to define an infinitesimal transformation strain as one that produces a vibration response that is qualitatively like that produced as $\gamma_0 \to 0$, and if one were to define a finite transformation strain as one that produces a vibration response that is qualitatively like that produce by a γ_0 that is close to or less than -0.5 or is close to or greater than 1, for the values of $\tilde{E}'$ and $\tilde{\nu}'$ considered, a transformation strain with a magnitude that is less than $|\gamma_0| \approx 0.06$ would be considered infinitesimal, and a transformation strain with a magnitude that is greater than $|\gamma_0| \approx 0.06$ would be considered finite. These "transition" values

$\gamma_0 \approx -0.06$ and $\gamma_0 \approx 0.06$ separating the infinitesimal transformation strains from the finite transformation strains are probably much smaller than most might have guessed beforehand. This also underlines the importance of treating a transformation strain that is not truly infinitesimal as a finite strain, for at least vibration problems.

9.2. Instabilities of the Numerical Routine

As mentioned previously, there are some instability problems with the numerical routine for values of $\tilde{\nu}$, $\tilde{E}$, and γ_0 outside of a certain region of the parameter space. In some cases, these instabilities resemble those of highly damped systems, which are sometimes referred to as *stiff systems* in the numerical methods literature [2]. In the remaining cases, the loss of stability of the numerical routine resembles that of a standard centered difference numerical routine for a wave equation when the coefficient multiplying the term representing the spatial derivative becomes greater than one. In both cases, however, the values of $\tilde{\nu}$, $\tilde{E}$, and γ_0 where the numerical routine loses stability depends on the values of T and h that are used. For example, for $h = 0.02$, $\tilde{E} = 1.15$ and $\gamma_0 = 0.1$, the numerical routine loses stability as $\tilde{\nu}$ is decreased at $\tilde{\nu} \approx 0.22$ when $T = 0.018$, and at $\tilde{\nu} \approx 0.09$ when $T = 0.01$. The loss of stability in both of these cases resembles that of stiff systems. In fact, for the latter case, the displacements reach their settling time in almost one half of one "cycle" of oscillation. One should note however that as $\tilde{\nu}$ gets close to zero, the assumption that ν is such that the magnitude of $\dot{s}^2$ is negligible in comparison to the magnitudes of the first-order terms in Equation (4.7) becomes less valid. For $h = 0.02$, $\tilde{\nu} = 0.5$, and $\gamma_0 = 0.1$, the value of $\tilde{E}$ beyond which the numerical routine is unstable is $\tilde{E} \approx 1.23$ when $T = 0.018$, and $\tilde{E} \approx 4.01$ when $T = 0.01$. Both of these values of $\tilde{E}$ correspond to $\tilde{E}(T/h)^2 \approx 1$ (recall that this term appears in the finite difference approximation of the forced wave equation for phase 2 given

by (8.12)). For $T = 0.018$, $h = 0.02$, $\tilde{E}' = 1.265$, and $\tilde{\nu}' = 0.05$, the numerical routing is stable for $0.021 < \gamma_0 < 0.152$. The loss of stability at $\gamma_0 \approx 0.021$ corresponds to $\tilde{E}(T/h)^2 \approx 1$, and the loss of stability at $\gamma_0 \approx 0.152$ resembles that of a stiff system. For $T = 0.01$, $h = 0.02$, $\tilde{E}' = 1.265$, and $\tilde{\nu}' = 0.05$, the numerical routing is stable for $-0.232 < \gamma_0 < 0.256$. For this case, the loss of stability at both $\gamma_0 \approx 0.256$ and $\gamma_0 \approx -0.232$ resembles that of stiff systems.

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